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Coherent states attached to Landau levels on the Poincaré disc

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Abstract

We construct a family of generalized coherent states attached to Landau levels of a charged particle moving in the Poincaré disc under a perpendicular uniform magnetic field. The corresponding coherent state transforms enable us to connect, by an integral transform, spaces of bound states of the particle with the space of square integrable functions on the real line. The established connection provides us with a new way to obtain hyperbolic Landau states.

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1. Introduction

Coherent states (CS) have attracted much attention in the recent decades. They are mathematical tools which provide a close connection between classical and quantum formalisms. In general, CS are a specific overcomplete family of vectors in the Hilbert space of the problem that describes the quantum phenomena. They are a useful tool in quantum theory and can be defined in several ways [1–4].

In quantum mechanics, considerable attention has been paid to the physics of a charged particle evolving in a plane under the influence of a perpendicular uniform magnetic field. This problem, called planar Landau problem [5], has been generalized to two-dimensional curved surfaces with a normal stationary magnetic field; see [6] and references therein.

In this paper, we are concerned with the hyperbolic Landau problem. We precisely deal with the Poincaré disc as configuration space and the Landau Hamiltonian on it. We construct for each hyperbolic Landau level a set of CS by following a generalization of the well-known canonical CS when written as series expansion in the basis of number states. The constructed CS belong to a weigthed Bergman space of analytic functions within the unit disc. The corresponding coherent state transforms enable us to connect, by an integral transform, spaces of bound states of the particle with the space of square integrable functions on the real line.

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The paper is organized as follows. In section 2, we present the formalism of constructing our coherent states. Section 3 deals with some required facts on the Landau problem on the Poincaré disc. In section 4, the formalism is applied so as to construct generalized coherent states attached to hyperbolic Landau levels. Section 5 is devoted to connect, by an integral transform, spaces of bound states of the particle with the space of square integrable functions on the real line. In section 6, we conclude with some remarks.

2. Generalized coherent states

Let (X, ν) be a measure space and let $\mathcal{A}^2 \subset L^2(X, \nu)$ be a closed subspace of infinite dimension. Let $\{\Phi_n\}_{n=1}^{\infty}$ be an orthogonal basis of $\mathcal{A}^2$ satisfying, for arbitrary $x \in X$,

$$\omega_\infty(x) := \sum_{n=1}^{\infty} \frac{|\Phi_n(x)|^2}{\rho_n} < +\infty,$$

where $\rho_n := \|\Phi_n\|^2_{L^2(X)}$. Define

$$K(x, y) := \sum_{n=1}^{\infty} \frac{\Phi_n(x)\overline{\Phi_n(y)}}{\rho_n}, \qquad x, y \in X.$$

Then, $K(x, y)$ is a reproducing kernel, $\mathcal{A}^2$ is the corresponding reproducing kernel Hilbert space and $\omega_\infty(x) = K(x, x)$, $x \in X$.

Let $\mathcal{H}$ be another Hilbert space with $\dim \mathcal{H} = \infty$ and $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Therefore, for $x \in X$, define

$$|x\rangle := (\omega_\infty(x))^{-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\Phi_n(x)}{\sqrt{\rho_n}} \phi_n. \tag{2.1}$$

The choice of the Hilbert space $\mathcal{H}$ defines a quantization of the set $X = \{x\}$ by the coherent states $|x\rangle$, via the inclusion map: $x \to |x\rangle$ from X into $\mathcal{H}$.

By definition, it is straightforward to show that $\langle x|x\rangle = 1$ and the coherent state transform $\mathcal{W} : \mathcal{H} \to \mathcal{A}^2 \subset L^2(X, \nu)$ defined by

$$\mathcal{W}[\phi](x) := (\omega_\infty(x))^{\frac{1}{2}} \langle x|\phi\rangle \tag{2.2}$$

is an isometry. Thus, for $\phi, \psi \in \mathcal{H}$, we have

$$\langle\phi|\psi\rangle_{\mathcal{H}} = \langle\mathcal{W}[\phi]|\mathcal{W}[\psi]\rangle_{L^2(X)} = \int_X \mathrm{d}\nu(x)\omega_\infty(x)\langle\phi|x\rangle\langle x|\psi\rangle$$

and thereby we have a resolution of the identity

$$\mathbf{1}_{\mathcal{H}} = \int_X \mathrm{d}\nu(x)\omega_\infty(x)|x\rangle\langle x|, \tag{2.3}$$

where $\omega_\infty(x)$ appears as a weight function. Note also that formula (2.1) can be considered as a generalization of the series expansion of the well-known canonical CS,

$$|\zeta\rangle = \left(\mathrm{e}^{|\zeta|^2}\right)^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \frac{\zeta^k}{\sqrt{k!}} \phi_k,$$

with $\phi_k := |k\rangle$ being the number states.

3. The Landau problem on the Poincaré disc

Let $\mathbf{D} = \{z \in \mathbb{C}, |z| < 1\}$ be the complex unit disc with the Poincaré metric

$$ds^2 = 4(1 - |z|^2)^{-2}\, dz\, d\overline{z}.$$

$\mathbf{D}$ is a complete Riemannian manifold with all sectional curvatures equal to -1. It has an ideal boundary $\partial\mathbf{D}$ identified with the circle $\{\omega \in \mathbb{C}, |\omega| = 1\}$. One refers to points $\omega \in \partial\mathbf{D}$ as points at infinity. The geodesic distance $d(z, w)$ between two points z and w is given by

$$\cosh d(z, w) = 1 + \frac{2|z - w|^2}{(1 - |z|^2)(1 - |w|^2)}.$$

By [6] the Schrödinger operator on $\mathbf{D}$ with a constant magnetic field of strength proportional to $|B|$ (Landau Hamiltonian) can be written as

$$L_B^{\mathbf{D}} := -(1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \overline{z}} - Bz(1 - |z|^2)\frac{\partial}{\partial z} + B\overline{z}(1 - |z|^2)\frac{\partial}{\partial \overline{z}} + B^2|z|^2. \tag{3.1}$$

For our purpose, we shall consider a slight modification of $L_B^{\mathbf{D}}$ given by the operator

$$H_B := 4L_B^{\mathbf{D}} - 4B^2$$

acting in the Hilbert space $L^{2,0}(\mathbf{D}) := L^2(\mathbf{D}, (1 - |z|^2)^{-2}\, d\mu(z))$ of complex-valued functions which are square integrable on $\mathbf{D}$ with respect to the measure $(1 - |z|^2)^{-2}\, d\mu(z)$, $d\mu(z)$ being the Lebesgue measure on $\mathbf{D}$. The spectrum of H_B in $L^{2,0}(\mathbf{D})$ consists of two parts: (i) a continuous part $[1, +\infty[$ which corresponds to scattering states, (ii) a finite number of eigenvalues E_m^B, $m \in \mathbb{Z}_+ \cap [0, |B| - \frac{1}{2}[$ (hyperbolic Landau levels) of the form

$$E_m^B := 4(|B| - m)(1 - |B| + m), \qquad B < -\tfrac{1}{2} \tag{3.2}$$

with infinite degeneracy. These eigenvalues below the continuous part of the spectrum have eigenfunctions which are called bound states since the particle in such a state cannot leave the system without additional energy. Note also that the operator H_B considered here is bounded below.

In order to present expressions of eigenfunctions of H_B in $L^{2,0}(\mathbf{D})$, we first write an intertwining relation between the operator H_B and a B-weight Maass Laplacian Δ_B as

$$H_B f(z) := 4\left(\frac{\overline{w} - \mathrm{i}}{w + \mathrm{i}}\right)^{-B} (-\Delta_B) \left(\frac{\overline{w} - \mathrm{i}}{w + \mathrm{i}}\right)^{B} f(\mathcal{C}(w)), \tag{3.3}$$

where f belongs to the space $C^\infty(\mathbf{D})$ of complex-valued functions which are infinitely differentiable on $\mathbf{D}$ and $z = \mathcal{C}(w) = (w - \mathrm{i})(w + \mathrm{i})^{-1} \in \mathbf{D}$, is the Cayley transform of w element of the Poincaré upper half-plane $\mathbf{H}^2 := \{\xi \in \mathbb{C}, \operatorname{Im}\xi > 0\}$. Explicitly,

$$\Delta_B = y^2\left(\partial_x^2 + \partial_y^2\right) - 2\mathrm{i}By\partial_x \tag{3.4}$$

with $C_0^\infty(\mathbf{H}^2)$ the space of complex-valued C^∞-functions with a compact support in $\mathbf{H}^2$ as its regular domain in the Hilbert space $L^2(\mathbf{H}^2, y^{-2}\, dx\, dy)$ of complex-valued functions which are square integrable on $\mathbf{H}^2$ with respect to the measure $y^{-2}\, dx\, dy$. According to (3.3), if $g : \mathbf{H}^2 \to \mathbb{C}$ is an eigenfunction of $-\Delta_B$ with λ as eigenvalue, then the function on the disc $\mathbf{D}$

$$z \mapsto \left(\frac{\overline{\mathcal{C}^{-1}(z)} - \mathrm{i}}{\mathcal{C}^{-1}(z) + \mathrm{i}}\right)^{|B|} g(\mathcal{C}^{-1}(z)), \qquad z \in \mathbf{D}$$

is an eigenfunction of H_B corresponding to the eigenvalue 4λ.

Let us denote by $\mathcal{A}_{B,m}^2(\mathbf{D})$ the eigenspace of H_B in $L^{2,0}(\mathbf{D})$, corresponding to the eigenvalue E_m^B given in (3.2), i.e.,

$$\mathcal{A}_{B,m}^2(\mathbf{D}) := \left\{\Phi : \mathbf{D} \to \mathbb{C}, \Phi \in L^{2,0}(\mathbf{D}) \text{ and } H_B\Phi = E_m^B\Phi\right\}.$$

Then by [7, p 288] (see also [8]) and relation (3.3), we obtain (up to multiplicative constants) the following orthogonal basis of $\mathcal{A}^2_{B,m}(\mathbf{D})$:

$$\Phi_j^{B,m}(z) := |z|^{|j|}(1-|z|^2)^{(|B|-m)}\,\mathrm{e}^{-ij\arg z} \times {}_2F_1\left(-m+\tfrac{1}{2}(j+|j|), 2|B|-m+\tfrac{1}{2}(|j|-j), 1+|j|; |z|^2\right),$$

$j \in \mathbb{Z}$ with $j \leqslant m$, where ${}_2F_1(a,b,c;x)$ is the Gauss hypergeometric function [9].

Also, we calculate the norm square of the eigenfunction $\Phi_j^{B,m}$ as

$$\begin{aligned}\rho_j^{B,m} &:= \left\|\Phi_j^{B,m}\right\|^2 = \int_{\mathbf{D}} \mathrm{d}\mu(z)(1-|z|^2)^{-2}\left|\Phi_j^{B,m}(z)\right|^2 \\ &= \int_{\mathbf{D}} \mathrm{d}\mu(z)|z|^{2|j|}(1-|z|^2)^{2(|B|-m)-2} \\ &\quad\times \left|{}_2F_1\left(-m+\frac{1}{2}(j+|j|), 2|B|-m+\frac{1}{2}(|j|-j), 1+|j|; |z|^2\right)\right|^2.\end{aligned}$$

Using polar coordinates $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant r < 1$ and $0 \leqslant \theta \leqslant 2\pi$, the above norm square reads

$$\begin{aligned}\left\|\Psi_j^{B,m}\right\|^2 &= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \mathrm{d}r r^{2|j|+1}(1-r^2)^{2(|B|-m)-2} \\ &\quad\times \left|{}_2F_1\left(-m+\frac{1}{2}(j+|j|), 2|B|-m+\frac{1}{2}(|j|-j), 1+|j|; r^2\right)\right|^2.\end{aligned}$$

We set $r^2 = \frac{1-t}{2}$ and we make use of the relation [9, p 1036]

$${}_2F_1\left(k+\alpha+\beta+1, -k, 1+\alpha; \frac{1-t}{2}\right) = \frac{k!\Gamma(1+\alpha)}{\Gamma(k+1+\alpha)} P_k^{(\alpha,\beta)}(t).$$

Connecting the hypergeometric function ${}_2F_1$ with the Jacobi polynomial $P_k^{(\alpha,\beta)}(t)$ [9], for the parameters

$$\alpha = |j|, \qquad \beta = 2(|B|-m)-1 \qquad \text{and} \qquad k = m - \tfrac{1}{2}(|j|+j),$$

we obtain that

$$\begin{aligned}\left\|\Psi_j^{B,m}\right\|^2 &= 2\pi\left(\frac{\Gamma(1+|j|)\left(m-\frac{1}{2}(|j|+j)\right)!}{\Gamma\left(m+1+\frac{1}{2}(|j|-j)\right)}\right)^2 \\ &\quad\times \int_0^1 \mathrm{d}r r^{2|j|+1}(1-r^2)^{2(|B|-m)-2}\left(P_{m-\frac{1}{2}(|j|+j)}^{(|j|,2(|B|-m)-1)}(t)\right)^2.\end{aligned}$$

We set $t = -u$ and we use the identity $P_k^{(\alpha,\beta)}(t) = (-1)^k P_k^{(\beta,\alpha)}(-t)$; then the last integral reads

$$I := 2^{-|j|-2|B|+2m}\int_{-1}^{+1}(1-u)^{2(|B|-m)-1-1}(1+u)^{|j|}\left(P_{m-\frac{1}{2}(|j|+j)}^{(2(|B|-m)-1,|j|)}(t)\right)^2 \mathrm{d}u.$$

Making use of the identity [9, p 842]

$$\int_{-1}^{+1}(1-u)^{\gamma-1}(1+u)^{\delta}\left(P_k^{(\gamma,\delta)}(u)\right)^2 \mathrm{d}u = \frac{2^{\gamma+\delta}\Gamma(\gamma+k+1)\Gamma(\delta+k+1)}{k!\gamma\Gamma(\gamma+\delta+k+1)}$$

$$\operatorname{Re}\gamma > 0, \qquad \operatorname{Re}\delta > -1$$

for the parameters

$$\gamma = 2(|B|-m)-1, \qquad \delta = |j| \qquad \text{and} \qquad k = m-\tfrac{1}{2}(|j|+j),$$

we obtain that

$$I = \frac{2^{-1}}{(2(|B|-m)-1)} \frac{\Gamma\left(m+\frac{1}{2}(|j|-j)+1\right)\Gamma\left(2|B|-m-\frac{1}{2}(|j|+j)\right)}{\Gamma\left(m-\frac{1}{2}(|j|+j)+1\right)\Gamma\left(2|B|-m+\frac{1}{2}(|j|-j)\right)}.$$

Finally,

$$\|\Phi_j^{B,m}\|^2 = \frac{\pi(\Gamma(1+|j|))^2}{(2(|B|-m)-1)} \frac{\Gamma\left(m-\frac{1}{2}(|j|+j)+1\right)\Gamma\left(2|B|-m-\frac{1}{2}(|j|+j)\right)}{\Gamma\left(m+\frac{1}{2}(|j|-j)+1\right)\Gamma\left(2|B|-m+\frac{1}{2}(|j|-j)\right)}.$$

Now, using the completeness relation ([7, p 288, equation (3.5)] for the Maass Laplacian Δ_B on $\mathbf{H}^2$,

$$\delta(\xi-\zeta) = \sum_{l\in\mathbb{Z}\cap[0,|B|-\frac{1}{2}[} \left(\frac{2|B|-2l-1}{4\pi}\right)\left(\frac{\xi-\overline{\zeta}}{\zeta-\overline{\xi}}\right)^B P_{|B|-l,B}(\cosh d(\xi,\zeta))$$

$$+\int_{\mathrm{Re}\, s=\frac{1}{2}} \mathrm{d}s\, \frac{\left(s-\frac{1}{2}\right)\sin 2\pi s}{8\pi \sin\pi(s-B)\sin\pi(s+B)}\left(\frac{\xi-\overline{\zeta}}{\zeta-\overline{\xi}}\right)^B P_{s,B}(\cosh d(\xi,\zeta)),$$

where

$$P_{s,B}(u) := \left(\frac{2}{u+1}\right)^s {}_2F_1\left(s-|B|, s+|B|, 1; \frac{u-1}{u+1}\right)$$

for $l=m$, we obtain the expression for the diagonal $K_{B,m}(z,z)$ of the reproducing kernel function $K_{B,m}(.,.)$ of the Hilbert eigenspace $\mathcal{A}^2_{B,m}(\mathbf{D})$ as $\omega_\infty(z) = \pi^{-1}(2|B|-2m-1)$.

4. Coherent states for hyperbolic Landau levels

Here, we specify the Hilbert space that will carry the generalized coherent states and fix an orthonormal basis in this space. We precisely take weighted Bergman space $\mathcal{H}_{(|B|-m)}$ of analytic functions ϕ on $\mathbf{D}$, which are of finite norm

$$\|\phi\|^2_{(|B|-m)} := \int_{\mathbf{D}} \mathrm{d}\mu(\xi)\Omega_{(|B|-m)}(\xi)|\phi(\xi)|^2 < +\infty,$$

$\Omega_{(|B|-m)}$ being the weight function given by

$$\Omega_{(|B|-m)}(\xi) := \pi^{-1}(2(|B|-m)-1)(1-|\xi|^2)^{2(|B|-m)-2},$$

with $(|B|-m)\in\left\{1,\frac{3}{2},2,\ldots\right\}$. The scalar product in $\mathcal{H}_{(|B|-m)}$ is written as

$$\langle\psi,\phi\rangle_{\mathcal{H}_{(|B|-m)}} = \int_{\mathbf{D}} \mathrm{d}\mu(\xi)\Omega_{(|B|-m)}(\xi)\psi(\xi)\overline{\phi(\xi)}.$$

Due to the rotational symmetry $\Omega_{|B|-m}(\xi) = \Omega_{|B|-m}(|\xi|)$, monomials $\xi\mapsto\xi^p$ with different degree $p\in\{0,1,2,\ldots\}$ are orthogonal, so the well-known reproducing kernel of $\mathcal{H}_{(|B|-m)}$,

$$K(\xi,\zeta) = (1-\xi\overline{\zeta})^{-2(|B|-m)}, \qquad \xi,\zeta\in\mathbf{D}, \tag{4.1}$$

is diagonalized in terms of the basis functions as $K(\xi,\zeta) = \sum_p c_p\xi^p(\overline{\zeta})^p$ with suitable normalization constants c_p. Indeed, with the help of the binomial formula

$$(1-x)^{-a} = \sum_{0\leqslant p<+\infty} \frac{\Gamma(a+p)}{\Gamma(a)\Gamma(1+p)}x^p,$$

the kernel function in (4.1) can be expanded into a series as

$$(1-\xi\overline{\zeta})^{-2(|B|-m)} = \sum_{0\leqslant p<+\infty} \frac{\Gamma(2(|B|-m)+p)}{\Gamma(2(|B|-m))\Gamma(1+p)}(\xi\overline{\zeta})^{p}.$$

This enables us to take, for our purpose, the orthonormal basis functions,

$$\phi_j(\xi) := \left(\frac{\Gamma(2|B|-m-j)}{\Gamma(2(|B|-m))\Gamma(1+m-j)}\right)^{\frac{1}{2}} \xi^{m-j}, \qquad -\infty < j \leqslant m, \quad \xi \in \mathbf{D}. \tag{4.2}$$

Remark 4.1. Note that the space $\mathcal{H}_{(|B|-m)}$ is the Bargmann's canonical carrier space for a unitary irreducible representation (positive discrete series) of the Lie group $SU(1,1)$.

We can now construct for each hyperbolic Landau level E_m^B given in (3.2) a set of generalized coherent states (GCS) according to formula (2.1) as

$$|z\rangle := (\omega_\infty(z))^{-\frac{1}{2}} \sum_{-\infty<j\leqslant m} \frac{\Phi_j^{B,m}(z)}{\sqrt{\rho_j^{B,m}}}\phi_j$$

with the following meaning.

- $(X,\nu) = (\mathbf{D}, (1-|z|^2)^{-2}\,\mathrm{d}\mu(z)), \mathbf{D} = \{z \in \mathbb{C}, |z| < 1\}, \mathrm{d}\nu(z) = (1-|z|^2)^{-2}\,\mathrm{d}\mu(z)$, $\mathrm{d}\mu(z)$ being the Lebesgue measure on $\mathbf{D}$
- $\mathcal{A}^2 := \mathcal{A}^2_{B,m}(\mathbf{D})$ denotes the eigenspace of H_B in $L^{2,0}(\mathbf{D})$.
- $\omega_\infty(z) = \pi^{-1}(2|B|-2m-1)$,
- $\Phi_j^{B,m}(z)$ are the eigenfunctions given in terms of the Gauss hypergeometric function ${}_2F_1$,
- $\rho_j^{B,m}$ being the norm square of $\Phi_j^{B,m}$,
- $\mathcal{H} := \mathcal{H}_{(|B|-m)}$ is the weighted Bergman space of index $(|B|-m)$
- ϕ_j are elements of the orthonormal basis of $\mathcal{H}_{(|B|-m)}$ given in (4.2).

The wavefunctions of these GCS are expressed as

$$\langle\xi|z,B,m\rangle := \frac{(1-|z|^2)^{|B|-m}}{\sqrt{\Gamma\left(2(|B|-m)\right)}} \sum_{-\infty\leqslant j\leqslant m} c_j^{B,m}\frac{|z|^{|j|}\,\mathrm{e}^{-ij\arg z}\xi^{m-j}}{|j|!}$$
$$\times\, {}_2F_1\left(-m+\frac{1}{2}(j+|j|), 2|B|-m+\frac{1}{2}(|j|-j), 1+|j|; |z|^2\right),$$

with

$$c_j^{B,m} := \left(\frac{\Gamma\left(m-\frac{1}{2}(|j|+j)+1\right)\Gamma\left(2|B|-m-\frac{1}{2}(|j|+j)\right)}{\Gamma\left(m+\frac{1}{2}(|j|-j)+1\right)\Gamma\left(2|B|-m+\frac{1}{2}(|j|-j)\right)}\frac{\Gamma(1+m-j)}{\Gamma(2|B|-m-j)}\right)^{-\frac{1}{2}}.$$

Remark 4.2. Note that for $m=0$, the wavefunctions of these GCS reduce to

$$\langle\xi|z,B,0\rangle = (1-|z|^2)^{|B|}\sum_{-\infty<j\leqslant 0}\frac{\Gamma(2|B|-j)}{\Gamma(2|B|)}\frac{\xi^{-j}|z|^{-j}\,\mathrm{e}^{-ij\arg z}}{(-j)!}.$$

Setting $-j=k$, then

$$\langle\xi|z,B,0\rangle = (1-|z|^2)^{|B|}\sum_{k=0}^{+\infty} z^k\left(\frac{\Gamma(2|B|+k)}{k!\Gamma(2|B|)}\right)^{\frac{1}{2}}|B,k\rangle,$$

where

$$\langle\xi|B,k\rangle = \sqrt{\frac{\Gamma(2|B|+k)}{k!\Gamma(2|B|)}}\xi^k,$$

is an orthonormal basis spanning the Hilbert space $\mathcal{H}_{|B|}$. It is clear that $\langle\xi|z,B,0\rangle$ are the wavefunctions of Perelomov's coherent states based on $SU(1,1)$ [3].

5. An integral transform

By (2.2) of section 1, the coherent state transform
$\mathcal{W}_{B,m}:\mathcal{H}_{(|B|-m)}\to\mathcal{A}^2_{B,m}(\mathbf{D})\subset L^{2,0}(\mathbf{D})$ defined by

$$\mathcal{W}_{B,m}[\phi](z):=(\omega_\infty(z))^{\frac{1}{2}}\langle m,B,z|\phi\rangle_{\mathcal{H}_{(|B|-m)}}$$

is an isometrical embedding. This constitutes a characterization of bound states of the particle. Moreover, we can exploit an integral transform T_γ connecting the weighted Bergman space $\mathcal{H}_\gamma$ of index γ and $L^2(\mathbb{R})$, which has the following integral kernel [10, p 3619]:

$$T_\gamma(\xi,x):=\frac{(1-|\xi|^2)^\gamma 2^{\gamma-\mathrm{i}x}}{\sqrt{2\pi(2\gamma-1)!}}\Gamma(\gamma-\mathrm{i}x)(1-\mathrm{i}x)^{-\gamma-\mathrm{i}x}(1+\mathrm{i}x)^{-\gamma+\mathrm{i}x}.$$

We precisely take $\gamma=|B|-m$ and we consider the integral transform

$$\mathcal{Z}_{B,m}:L^2(\mathbb{R})\to\mathcal{A}^2_{B,m}(\mathbf{D})$$

defined by $\mathcal{Z}_{B,m}:=\mathcal{W}_{B,m}\circ T_{(|B|-m)}$ which connect $L^2(\mathbb{R})$ and the space $\mathcal{A}^2_{B,m}(\mathbf{D})$ of bound states of the particle. We note that it would be of interest to obtain a closed-form of the integral kernel of the transform $\mathcal{Z}_{B,m}$.

6. Concluding remarks

For a charged particle moving in the Poincaré disc under the influence of a perpendicular uniform magnetic field, we have constructed for each Landau level a set of generalized coherent states as series expansion in a basis of analytic functions belonging to a weighted Bergman space associated with the unit disc. For the lowest Landau level, these coherent states reduce to Perelomov's coherent states based on the Lie group $SU(1,1)$. Under the corresponding coherent state transforms, bound states of the particle are images of analytic functions within the unit disc. On the other hand, these coherent state transforms enable us to connect, by an integral transform, spaces of bound states of the particle with the space of square integrable functions on the real line. The established connection provides us with a new way of looking at Landau states on the hyperbolic disc.

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